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Directed polymers on trees: a martingale approach

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Abstract. We use martingale methods and simple convexity arguments to compute rigorously the limiting free energy in the problem of directed polymers on a tree. The limit is a degenerate random variable and convergence holds almost surely. The only assumption on the common distribution of the random potentials attached to the bonds of the tree is that its Laplace transform exists everywhere in $[0, \infty)$.

1. Introduction

Directed polymers in random media have received much attention in recent years [1–8]. The problem can be described as follows: a directed random walk takes place on a regular lattice; independent identically distributed energies are attached to each bond of the lattice, and paths are given a Gibbs weight corresponding to the sum of the energies of the visited bonds. The main object of interest is the effect of the disorder on the asymptotic properties of the walk; typically, one expects a transition from a diffusive regime at high temperature to a superdiffusive behaviour at low temperature (see [3] and references quoted therein).

Because the above problem remains largely open as far as rigorous derivations are concerned, it is of interest to consider simplified models. The case where the lattice is replaced by a Cayley tree is sufficiently rich to give rise to a phase transition (in this model it is the free energy rather than the mean square displacement which is the central object). It has been studied by various heuristic methods such as the replica argument [7], an extrapolation of the properties of the generalized random energy model [6] and an analogy with known properties of branching diffusions [2]. Our treatment, which is based on martingales, has the advantage of being both rigorous and transparent. It is only fair to point out that the calculation of the ground state energy of our model (i.e. the zero-temperature limit of the free energy) is substantially the same as a question known in the theory of branching processes as the first birth problem, and solved in [9].

Apart from its connection with the original directed walk question, the tree problem can also be seen as a generalization of the random energy model introduced in [10] as a caricature of spin glasses and solved rigorously in [11, 12]; in this last model the energies of different *paths* are independent, in sharp contrast with the tree problem. It is all the more remarkable that our approach produces a solution which is rather less intricate than either [11] or [12].

2. Description of the problem

Consider a Cayley tree with branching ratio two; label the bonds of the tree by two integers (j,k) where j identifies the generation and $k \in \{1,\dots,2^j\}$ numbers the bonds from left to right within the jth generation.

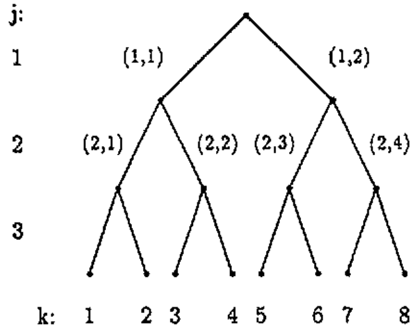

Figure 1. Labelling the bonds of the tree.

A *path* w starting at the top of the tree and of length $|w| = n$ is a finite sequence $\{(j, w_j), 1 \leqslant j \leqslant n\}$ obeying the constraint $w_{j+1} = 2w_j + s_j$, where the numbers $s_j \in \{-1, 0\}$ correspond to taking the left or right branch out of generation j.

Attach independent identically distributed random variables $V_{j,k}$ to the bonds of the tree. The only assumption that we make on the common distribution of the $V_{j,k}$ is that their negative part falls off sufficiently fast to ensure that the function

$$\phi(\beta) = E[\mathrm{e}^{-\beta V}] \tag{1}$$

exists for all $\beta \geqslant 0$. The infinite differentiability of $\phi(\beta)$ follows from this assumption.

The central object of our investigations is the (random) *partition function*

$$Z_n(\beta) = \sum_{w:|w|=n} \mathrm{e}^{-\beta \sum_{j=1}^{n} V_{j,w_j}} \tag{2}$$

and in particular the large n limit of the *free energy density* $(1/\beta n) \log Z_n(\beta)$.

3. The main results

Note that even though the random variables $V_{j,k}$ are mutually independent, the exponents $\sum_{j=1}^{n} V_{j,w_j}$ and $\sum_{j=1}^{n} V_{j,w'_j}$ in (2) are not in general independent for different paths w, w'. Thus, in contrast to [11, 12] the partition function is not a sum of independent random variables. However the dependence between the summands is of a very special type; let

$$\mathcal{V}^n = \{V_{j,k}, 1 \leqslant k \leqslant 2^j, 1 \leqslant j \leqslant n\} \tag{3}$$

denote the set of all the random variables $V_{j,k}$ between generations 1 and n. Define

$$M_n(\beta) = Z_n(\beta)/(2\phi(\beta))^n. \tag{4}$$

Then we have:

Proposition 1. The sequence $\{M_n(\beta), n \geqslant 1\}$ is a *martingale* with respect to the increasing family of random variables $\{\mathcal{V}^n, n \geqslant 1\}$, that is to say

$$E[M_{n+1}(\beta)|\mathcal{V}^n] = M_n(\beta) \tag{5}$$

where the left-hand side is the *conditional expectation* of $M_{n+1}(\beta)$ *given* all the random variables in $\mathcal{V}_n$.

Proof.

$$Z_{n+1}(\beta) = \sum_{w:|w|=n} \mathrm{e}^{-\beta \sum_{j=1}^{n} V_{j,w_j}} \sum_{s_n=-1,0} \mathrm{e}^{-\beta V_{n+1,2w_n+s_n}} \tag{6}$$

so that

$$E[Z_{n+1}(\beta)|\mathcal{V}^n] = \sum_{w:|w|=n} \mathrm{e}^{-\beta \sum_{j=1}^{n} V_{j,w_j}} E\left[\sum_{s_n=-1,0} \mathrm{e}^{-\beta V_{n+1,2w_n+s_n}}\right] \tag{7}$$

$$= 2\phi(\beta) Z_n(\beta). \tag{8}$$

Divide by $(2\phi(\beta))^{n+1}$ to obtain the result. Note that $E[M_n(\beta)] = 1$. □

Remark. It is fairly common for the normalized partition function of a random system to be a martingale, see [4, 13]. This property is usually of limited value unless it is accompanied by boundedness of some moment of order larger than one (or more generally uniform integrability). The proof of such a bound is highly model-dependent and constitutes the core of any study of a random system by the method of martingales, see proposition 2.

Since $M_n(\beta)$ is a positive martingale, it converges almost surely to a finite random variable $M_\infty(\beta)$, see [14]. But, noting that

$$\frac{1}{\beta n} \log Z_n(\beta) = \frac{1}{\beta n} \log[(2\phi(\beta))^n M_n(\beta)] \tag{9}$$

$$= \frac{1}{\beta} \log[2\phi(\beta)] + \frac{1}{\beta n} \log M_n(\beta) \tag{10}$$

we see that

$$\limsup_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) \leqslant \frac{1}{\beta} \log[2\phi(\beta)] \quad \text{a.s.} \tag{11}$$

and moreover, if $M_\infty(\beta)$ is strictly positive with probability one

$$\lim_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) = \frac{1}{\beta} \log[2\phi(\beta)] \quad \text{a.s.}. \tag{12}$$

In common with other limits of non-negative polymer-related martingales (see [4, 13]) the limit random variable is either strictly positive or concentrated at zero.

Lemma 1. For any fixed $\beta \geqslant 0, P[M_\infty(\beta) = 0]$ is equal to either zero or one.

Proof. Let L_n (respectively R_n) denote the set of paths of length n which start with a branch in the left (respectively right) direction. From the formula

$$M_n(\beta) = \mathrm{e}^{-\beta V_{11}} \sum_{w \in L_n} \mathrm{e}^{-\beta \sum_{j=2}^{n} V_{j,w_j}} [2\phi(\beta)]^{-n} + \mathrm{e}^{-\beta V_{12}} \sum_{w \in R_n} \mathrm{e}^{-\beta \sum_{j=2}^{n} V_{j,w_j}} [2\phi(\beta)]^{-n} \tag{13}$$

it is clear that the event $\{\lim_{n\to\infty} M_n(\beta) = 0\}$ is independent of V_{11} and V_{12}; one can see in the same way that it is independent of $\mathcal{V}_p$ for every p. The result follows from Kolmogorov's zero–one law [14]. □

Remark. The above argument cannot be applied to show that $P[M_\infty(\beta) > x]$ is either 0 or 1; in general the random variable $M_\infty(\beta)$ is *not* degenerate.

Using lemma 1, it suffices to know that $E[M_\infty(\beta)] > 0$ to conclude that $P[M_\infty(\beta) = 0] = 0$, and consequently that (12) holds. Hence the logic of the rest of the proof: show that, in the appropriate range of values of β, $M_n(\beta)$ has a bounded moment of order larger than one, i.e.

$$\sup_{n\geqslant 1} E[M_n^\alpha(\beta)] < \infty \qquad \text{for some } \alpha > 1. \tag{14}$$

This will ensure that $M_n(\beta)$ is uniformly integrable and thus that it converges to $M_\infty(\beta)$ in L^1, implying $E[M_\infty(\beta)] = 1$ and thus ruling out $M_\infty(\beta) = 0$. We start by computing the second moment.

Lemma 2.

$$E[M_{n+1}^2(\beta)|\mathcal{V}^n] = M_n^2(\beta) + \lambda(\beta)[\phi(2\beta)/2\phi^2(\beta)]^n M_n(2\beta) \tag{15}$$

where $\lambda(\beta)$ is the non-negative function

$$\lambda(\beta) = \frac{\phi(2\beta) - \phi^2(\beta)}{2\phi^2(\beta)}. \tag{16}$$

Proof. This is a straightforward calculation along the lines of proposition 1. □

Remark. Taking expectations across formula (15) and summing over n, it follows that

$$\sup_{k\geqslant 1} E[M_k^2(\beta)] < \infty \qquad \text{whenever } \phi(2\beta) < 2\phi^2(\beta). \tag{17}$$

This provides a sufficient condition for the uniform integrability of $M_n(\beta)$; however, this condition falls short of the optimal result that we will derive in proposition 2 using the following elementary observation:

Lemma 3. For any real numbers $x_j, 1 \leqslant j \leqslant n$ the function $(\sum_{j=1}^{n} \mathrm{e}^{-\beta x_j})^{1/\beta}$ is decreasing in $\beta, (\beta \geqslant 0)$.

Proof. Obviously

$$\mathrm{e}^{-\beta x_j} \Big/ \sum_{k=1}^{n} \mathrm{e}^{-\beta x_k} \leqslant 1 \tag{18}$$

so that if $\beta' \geqslant \beta$

$$\left(\mathrm{e}^{-\beta x_j} \Big/ \sum_{k=1}^{n} \mathrm{e}^{-\beta x_k}\right)^{\beta'/\beta} \leqslant \mathrm{e}^{-\beta x_j} \Big/ \sum_{k=1}^{n} \mathrm{e}^{-\beta x_k}. \tag{19}$$

Sum over j and use elementary manipulations to obtain

$$\left(\sum_{j=1}^{n} \mathrm{e}^{-\beta' x_j}\right)^{1/\beta'} \leqslant \left(\sum_{j=1}^{n} \mathrm{e}^{-\beta x_j}\right)^{1/\beta}. \tag{20}$$

□

Proposition 2. Define

$$f(\beta) = \frac{1}{\beta} \log[2\phi(\beta)]. \tag{21}$$

For every β such that $f'(\beta) < 0$, there exists $\alpha > 1$ such that $\sup_{n \geqslant 1} \boldsymbol{E}[M_n^{\alpha}(\beta)] < \infty$.

Proof. Take $0 < \alpha < 2$. Using Jensen's inequality and lemma 2 we have

$$\begin{aligned}
\boldsymbol{E}[M_{n+1}^{\alpha}(\beta)|\mathcal{V}^n] &\leqslant (\boldsymbol{E}[M_{n+1}^{2}(\beta)|\mathcal{V}^n])^{\alpha/2} && (22)\\
&= (M_n^2(\beta) + \lambda(\beta)[\phi(2\beta)/2\phi^2(\beta)]^n M_n(2\beta))^{\alpha/2} && (23)\\
&\leqslant M_n^{\alpha}(\beta) + \lambda^{\alpha/2}(\beta)[\phi(2\beta)/2\phi^2(\beta)]^{n\alpha/2} M_n^{\alpha/2}(2\beta). && (24)
\end{aligned}$$

But by lemma 3

$$Z_n^{1/2\beta}(2\beta) \leqslant Z_n^{1/\alpha\beta}(\alpha\beta) \quad \text{a.s.} \tag{25}$$

so that

$$M_n^{\alpha/2}(2\beta) \leqslant M_n(\alpha\beta)[2\phi(\alpha\beta)]^n / [2\phi(2\beta)]^{n\alpha/2} \tag{26}$$

implying

$$\boldsymbol{E}[M_{n+1}^{\alpha}|\mathcal{V}^n] \leqslant M_n^{\alpha}(\beta) + \lambda^{\alpha/2}(\beta)[2\phi(\alpha\beta)/(2\phi(\beta))^{\alpha}]^n M_n(\alpha\beta). \tag{27}$$

Taking expectations across and summing over n we get

$$E[M_k^\alpha(\beta)] \leqslant E[M_1^\alpha(\beta)] + \lambda^{\alpha/2}(\beta) \sum_{n=1}^{k-1} [2\phi(\alpha\beta)/(2\phi(\beta))^\alpha]^n. \quad (28)$$

This shows that

$$\sup_{k \geqslant 1} E[M_k^\alpha(\beta)] < \infty \quad (29)$$

whenever

$$2\phi(\alpha\beta) < (2\phi(\beta))^\alpha. \quad (30)$$

To complete the proof, it suffices to note that

$$2\phi(\alpha\beta)/(2\phi(\beta))^\alpha = \exp[\alpha\beta(f(\alpha\beta) - f(\beta))] \quad (31)$$

so that if $f'(\beta) < 0$ there exists $\alpha > 1$ such that (30) holds. The bound (29) with $\alpha > 1$ is well known to imply uniform integrability of $M_n(\beta)$ [14]. □

In view of the above result, we need to characterize the possible shapes of the graph of $f(\beta)$.

Lemma 4. Either there exists $\beta_c > 0$ such that the function $f(\beta)$ is strictly decreasing on $(0, \beta_c)$ and strictly increasing on (β_c, ∞), or the function $f(\beta)$ is strictly decreasing on $(0, \infty)$.

Proof. If the random variable V is concentrated at a point, $f(\beta)$ is trivially strictly decreasing. In all other cases, $\beta f(\beta)$ is strictly convex, so that for every β, β_0 with $\beta \neq \beta_0$

$$\beta f(\beta) > \beta_0 f(\beta_0) + [f(\beta_0) + \beta_0 f'(\beta_0)][\beta - \beta_0]. \quad (32)$$

In particular, if f has a local extremum at β_c

$$\beta f(\beta) > \beta f(\beta_c) \qquad \text{for all } \beta \neq \beta_c \quad (33)$$

so that β_c is the unique value where f achieves its global minimum. Finally, in the absence of a local extremum, f is strictly monotonic; since $f(\beta) \to \infty$ as $\beta \to 0$, f must be strictly decreasing. □

Remarks. (i) We will denote by β_c the value at which $f(\beta)$ takes its minimum, with the convention $\beta_c = \infty$ if no such local minimum exists.
(ii) As an illustration of the lemma consider the case where V is exponentially distributed with parameter λ; we have in this case

$$\phi(\beta) = \frac{\lambda}{\beta + \lambda} \quad (34)$$

$$f(\beta) = \frac{1}{\beta} \log \frac{2\lambda}{\beta + \lambda} \quad (35)$$

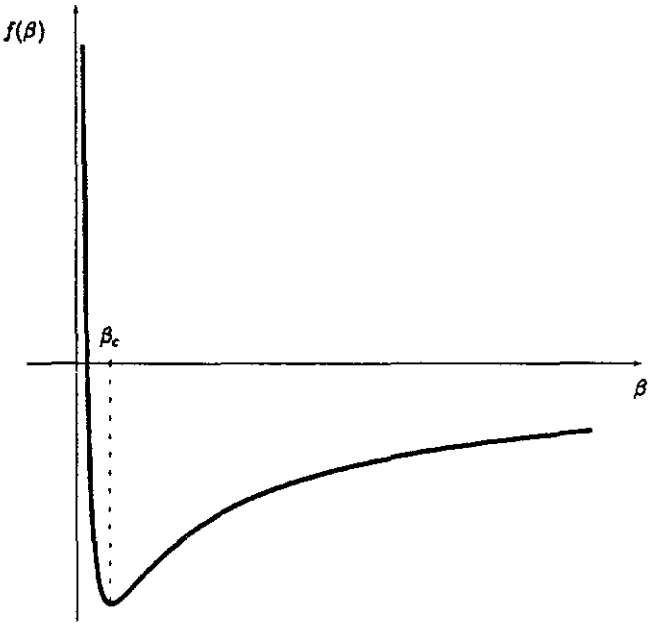

Figure 2. The function $f(\beta)$ when V is exponentially distributed.

and β_c is the unique solution of

$$\log \frac{2\lambda}{\beta+\lambda} = -\frac{\beta}{\beta+\lambda}. \tag{36}$$

The graph of $f(\beta)$ is shown in figure 2.

In preparation for the main theorem, we note the following simple consequence of lemma 3:

Lemma 5. For any real numbers $x_j, 1 \leqslant j \leqslant n$, the function

$$g(\beta) = \frac{1}{\beta} \log \sum_{j=1}^{n} \mathrm{e}^{-\beta x_j} \qquad \beta \geqslant 0$$

is decreasing and convex in β.

Proof. Decreasingness follows from lemma 3. Moreover

$$\beta g'(\beta) = -g(\beta) - \sum x_j \mathrm{e}^{-\beta x_j} \Big/ \sum \mathrm{e}^{-\beta x_j} \tag{37}$$

$$\beta g''(\beta) = -2g'(\beta) + \sum x_j^2 \mathrm{e}^{-\beta x_j} \Big/ \sum \mathrm{e}^{-\beta x_j} - \left(\sum x_j \mathrm{e}^{-\beta x_j} \Big/ \sum \mathrm{e}^{-\beta x_j} \right)^2 \geqslant 0 \tag{38}$$

proving convexity. $\square$

We can now state and prove the main result of this article:

Theorem 1. The following limit holds almost surely

$$\lim_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) = \begin{cases} f(\beta) & \beta \leqslant \beta_c \\ f(\beta_c) & \beta > \beta_c \end{cases} \tag{39}$$

where β_c is defined as in the last remark.

Proof. (i) When $\beta < \beta_c$, $f'(\beta) < 0$ so that proposition 2 is valid; hence $\boldsymbol{E}[M_\infty(\beta)] = 1$ and so, using lemma 1, the result follows as in (10), (12).

What we have just shown can be restated as follows; define

$$\Omega_\beta = \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) = f(\beta)\right\}. \tag{40}$$

Then

$$\boldsymbol{P}[\Omega_\beta] = 1 \qquad \text{whenever } 0 < \beta < \beta_c. \tag{41}$$

For the second part of the proof we need the stronger result

$$\boldsymbol{P}[\bigcap_{0<\beta<\beta_c} \Omega_\beta] = 1 \tag{42}$$

which can be proved as follows: first consider a *countable dense* set I in $(0, \beta_c)$. It follows clearly from (42) that

$$\boldsymbol{P}[\bigcap_{\beta\in I} \Omega_\beta] = 1. \tag{43}$$

Next consider an arbitrary $\omega \in \cap_{\beta\in I}\Omega_\beta$; for any $\beta_0 \in (0, \beta_c)$ construct sequences $\beta_k^+ \searrow \beta_0$ and $\beta_k^- \nearrow \beta_0, \beta_k^+, \quad \beta_k^- \in I$. For any ω in $\cap_{\beta\in I}\Omega_\beta$ we deduce from lemma 5

$$\limsup_{n\to\infty} \frac{1}{\beta_0 n} \log Z_n(\beta_0)(\omega) \leqslant f(\beta_k^-) \tag{44}$$

$$\liminf_{n\to\infty} \frac{1}{\beta_0 n} \log Z_n(\beta_0)(\omega) \geqslant f(\beta_k^+). \tag{45}$$

Let $k \nearrow \infty$ and conclude that $\omega \in \Omega_{\beta_0}$. Thus $\cap_{\beta\in I}\Omega_\beta = \cap_{0<\beta<\beta_c}\Omega_\beta$ and (42) follows from (43).

(ii) When $\beta \geqslant \beta_c$ we have no guarantee that $M_n(\beta)$ is uniformly integrable, so that the above method fails; in fact it turns out that $M_\infty(\beta) = 0$ when $\beta > \beta_c$, see the remark following this proof. However, using the decreasingness in lemma 5, we have for every $\varepsilon > 0$

$$\frac{1}{\beta n} \log Z_n(\beta) \leqslant \frac{1}{(\beta_c - \varepsilon)n} \log Z_n(\beta_c - \varepsilon) \quad \text{a.s.} \tag{46}$$

so that using (i) we have

$$\limsup_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) \leqslant f(\beta_c - \varepsilon) \quad \text{a.s.}. \tag{47}$$

On the other hand, using the convexity result in lemma 5 we have, for every $\varepsilon > 0$

$$\frac{1}{\beta n} \log Z_n(\beta) \geqslant Y_n(\beta_c - \varepsilon)(\beta - \beta_c + \varepsilon) + \frac{1}{(\beta_c - \varepsilon)n} \log Z_n(\beta_c - \varepsilon) \tag{48}$$

where

$$Y_n(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{\beta n} \log Z_n(\beta)\right). \tag{49}$$

By (i), for almost every sample point ω in Ω the sequence of convex functions

$$\frac{1}{\beta n} \log Z_n(\beta)(\omega) \qquad \beta \leqslant \beta_c \tag{50}$$

converges to the *differentiable* function $f(\beta)$; hence their derivatives converge to $f'(\beta)$, so that (47) implies

$$\liminf_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) \geqslant f'(\beta_c - \varepsilon)(\beta - \beta_c + \varepsilon) + f(\beta_c - \varepsilon) \quad \text{a.s.}. \tag{51}$$

Noting that ε is arbitrary in (47) and (51) and that $\lim_{\varepsilon\to 0} f'(\beta_c - \varepsilon) = 0$, the result follows. □

Remark. When $\beta > \beta_c$

$$\lim_{n\to\infty} \frac{1}{\beta n} \log Z_n(\beta) < f(\beta) \quad \text{a.s.} \tag{52}$$

so that by (12), $M_\infty(\beta)$ must have a non-vanishing probability of being equal to zero and is thus concentrated at zero by lemma 1. In fact when $\beta \geqslant \beta_c$ theorem 1 implies

$$\lim_{n\to\infty} \frac{1}{\beta n} \log M_n(\beta) = f(\beta_c) - f(\beta) \quad \text{a.s.} \tag{53}$$

which shows that $M_n(\beta)$ converges to zero exponentially fast when $\beta > \beta_c$.

Convergence holds also in L^p if we supplement our standing assumption of finiteness of $E[\mathrm{e}^{-\beta V}]$, $\beta \geqslant 0$ with that of the existence of appropriate moments of V; the following proof is adapted from that of the corresponding result for the random energy model in [12]:

Theorem 2. The limit of theorem 1 holds also in L^p whenever $E[|V|^{p+\varepsilon}] < \infty$ for some $\varepsilon > 0$, $p \geqslant 1$.

Proof. As is well known (see [15]), it suffices to check that for fixed β, p the random variables $|(1/\beta n)\log Z_n(\beta)|^p$ are uniformly integrable. Denote by E_n^0 the *ground state energy*

$$E_n^0 = \min\{E_w, |w| = n\} \tag{54}$$

where

$$E_w = \sum_{j=1}^{n} V_{j,w_j} \tag{55}$$

is the energy of the path w.

Because of the obvious inequalities

$$-\frac{1}{n}E_n^0 \leqslant \frac{1}{\beta n}\log Z_n(\beta) \leqslant \frac{\log 2}{\beta} - \frac{1}{n}E_n^0 \tag{56}$$

it suffices to prove that for fixed p, $((1/n)E_n^0)^p$ are uniformly integrable random variables, that is to say

$$\lim_{\alpha\to\infty} \sup_n \boldsymbol{E}[|(1/n)E_n^0|^p; |(1/n)E_n^0|^p > \alpha] = 0. \tag{57}$$

In (57) we made use of the notation

$$\boldsymbol{E}[X; A] = \boldsymbol{E}[XI_A] \tag{58}$$

where X is a random variable and A an event with indicator function I_A. Note that if X and α are positive

$$\boldsymbol{E}[X; X > \alpha] = \int_\alpha^\infty \boldsymbol{P}[X > x]\,\mathrm{d}x + \alpha\boldsymbol{P}[X > \alpha]. \tag{59}$$

In order to prove (57), write

$$\begin{aligned}\boldsymbol{E}[|(1/n)E_n^0|^p; |(1/n)E_n^0|^p > \alpha]&\\ &= \boldsymbol{E}[((1/n)E_n^0)^p; (1/n)E_n^0 > \alpha^{1/p}]\\ &\quad + \boldsymbol{E}[(-(1/n)E_n^0)^p; -(1/n)E_n^0 > \alpha^{1/p}].\end{aligned} \tag{60}$$

The first term satisfies (57) because for any path w of length n, $E_n^0 \leqslant E_w$ so that

$$\left(\frac{1}{n}E_n^0\right)^p \leqslant \left(\frac{1}{n}E_w\right)^p \qquad \text{on } \left\{\frac{1}{n}E_n^0 > \alpha^{1/p}\right\} \tag{61}$$

and $((1/n)E_w)^p$ is uniformly integrable because

$$\sup_n \boldsymbol{E}\left[\left|\frac{1}{n}E_w\right|^{p+\varepsilon}\right] < \infty. \tag{62}$$

As for the second term in (60), it can be rewritten as follows by (59)

$$\int_\alpha^\infty P\left[\left(-\frac{1}{n}E_n^0\right)^p > x\right] \mathrm{d}x + \alpha P\left[\left(-\frac{1}{n}E_n^0\right)^p > \alpha\right]$$
$$= \int_\alpha^\infty P[E_n^0 \leqslant -nx^{1/p}]\,\mathrm{d}x + \alpha P[E_n^0 \leqslant -n\alpha^{1/p}]. \qquad (63)$$

But note that

$$P[E_n^0 \leqslant a] = P[\cup_{w:|w|=n}\{E_w \leqslant a\}] \leqslant 2^n P[E_w \leqslant a]$$
$$\leqslant 2^n \mathrm{e}^a E[\mathrm{e}^{-E_w}] = (2E[\mathrm{e}^{-V}])^n \mathrm{e}^a. \qquad (64)$$

Hence (63) is bounded above by

$$(2E[\mathrm{e}^{-V}])^n \left(\int_\alpha^\infty \mathrm{e}^{-nx^{1/p}}\,\mathrm{d}x + \alpha \mathrm{e}^{-n\alpha^{1/p}}\right)$$
$$\leqslant (2E[\mathrm{e}^{-V}]\mathrm{e}^{-(\alpha^{1/p})/2})^n \left(\int_\alpha^\infty \mathrm{e}^{-(nx^{1/p})/2}\,\mathrm{d}x + \alpha \mathrm{e}^{-(n\alpha^{1/p})/2}\right). \qquad (65)$$

The above expression attains its maximum over n at $n = 1$ for α large enough, and this maximum value tends obviously to zero when α tends to infinity. This completes the proof of (57) and of the theorem. □

Remark. As all the results in this article, the above theorem holds for trees with arbitrary branching ratio K provided that the definition (21) of f is replaced by $f(\beta) = (1/\beta)\log[K\phi(\beta)]$.

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